

A DUAL RECIPROCAL BOUNDARY ELEMENT FORMULATION FOR VISCOUS FLOWS

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SUMMARY

The advantages inherent in the Boundary Element Method (BEM) for potential flows are exploited to solve viscous flow problems. The trick is the introduction of a so-called Dual Reciprocal technique in which the convective terms are represented by a global function whose unknown coefficients are determined by collocation. The approach, which is necessarily iterative, converts the governing partial differential equations (PDEs) into integral equations via the distribution of fictitious sources or dipoles of unknown strength on the boundary. These integral equations consist of two parts. The first is a boundary integral term, whose kernel is the unknown strength of the fictitious sources and the fundamental solution of a convection-free flow problem. The second part is a domain integral term whose kernel is the convective portion of the governing PDEs. The domain integration can be transformed to the boundary by using the Dual Reciprocal (DR) concept. The resulting formulation is a pure boundary integral computational process.

INTRODUCTION

The major advantage the BEM approach enjoys over other techniques is the confinement of the computation to the boundary. The result is the reduction in the effective dimension of the problem. The efficiency with which linear problems in continua can be solved using BEM has received considerable mention in the literature during the past decade. Apart from the reduced dimensionality and the need for no special domain discretization, other derived advantages include:

1. the ability to handle infinitely large domains;
2. a much reduced coefficient matrix;
3. the ease with which singularities are handled;
4. the restriction of the discretization errors to the boundary, so that the solution is as good as the description of the boundary geometry;
5. the robustness when complex geometries are involved;
6. the ability to find solutions *a posteriori* at desired points, not at nodes predetermined by the domain discretization;

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7. the great latitude in solving transient problems by a) using the appropriate time dependent fundamental solution in the formulation; b) applying the technique in a transform domain (*e.g.*, Laplace or Fourier); or c) using a time marching procedure.

Efforts at applying boundary integral techniques to nonlinear problems are quite recent. For over a decade the main focus was largely on linear problems such as potential flows (see *e.g.*, Brebbia *et al.*, [1984], Liggett & Liu [1984]). BEM formulations for nonlinear (*e.g.*, Lafe *et al.*, [1981]; Lafe & Cahan [1990]) or those in heterogeneous continua (*e.g.*, Cheng [1984]; Lafe *et al.*, [1987-1992]) have relied heavily on iterative methods which still require some domain integration. The *Dual Reciprocal* technique creates a major path for exploiting the advantages of BEM to solve nonlinear problems such as convective flows. No domain integration is involved when the *Dual Reciprocal* approach is followed.

The original credit for *Dual Reciprocal* BEM concept goes to Nardini & Brebbia [1982] who first suggested an innovative approach for transforming domain integrals to the boundary. However, until recently, prior investigations (see *e.g.*, Brebbia [1991]; Partridge *et al.*, [1992]) did not make use of a complete set of global functions. A series of local radial functions were utilized. This made convergence difficult or impossible for a class of nonlinear problems. This author and his co-workers (see Cheng *et al.*, [1993]) have recently derived a set of complete coordinate functions which have been tested on a family of strongly nonlinear PDEs. Excellent results have been obtained with the complete set. This work opens the door to the application of BEM to a wide spectrum of complex flow problems.

In this paper, we present the full formulation of the Dual Reciprocal Boundary Element Method (DRBEM), for incompressible convective flows.

GOVERNING EQUATIONS

Let the flow region is represented by Ω and the boundary is Γ . The pertinent flow equations are:

- *Continuity Equation*

$$\nabla \cdot \mathbf{v} = 0 \quad (1)$$

- *Conservation of Momentum*

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \boldsymbol{\tau} + \mathbf{g} \quad (2)$$

where \mathbf{v} is the velocity, p is pressure, \mathbf{g} is the gravitational accelerator vector, $\boldsymbol{\tau}$ is the viscous stress tensor. If μ is viscosity, then for a Newtonian fluid, $\boldsymbol{\tau}$ is expressible in the form:

$$\boldsymbol{\tau} = \mu \nabla \mathbf{v}$$

Dimensionless Equations

Let L = characteristic length scale, \bar{v} = mean velocity, and η is the elevation of the point (\mathbf{x}). We can define the following dimensionless variables:

$$\mathbf{x}_* = \mathbf{x}/L_2 \quad (3)$$

$$\mathbf{v}_* = \mathbf{v}/\bar{v} \quad (4)$$

$$p_* = (p + g\eta)/(\rho\bar{v}^2) \quad (5)$$

$$t_* = t\bar{v}/L \quad (6)$$

With these, the above conservation statements can be made dimensionless:

$$\nabla \cdot \mathbf{v}_* = 0 \quad (7)$$

$$\frac{\partial \mathbf{v}_*}{\partial t_*} + (\mathbf{v}_* \cdot \nabla) \mathbf{v}_* = -\nabla p_* + \frac{1}{R_e} \nabla^2 \mathbf{v}_* \quad (8)$$

where

$$R_e = \rho\bar{v}L/\mu = \text{Reynolds Number} \quad (9)$$

The governing equations can be rearranged and written in the pseudo-Poisson form:

$$\nabla^2 \Phi(\mathbf{x}_*, t_*) = F(\mathbf{x}_*, t_*) \quad (10)$$

where

$$\Phi = \begin{cases} \mathbf{v}_* & \text{Velocity} \\ p_* & \text{Pressure} \end{cases} \quad (11)$$

and

$$F = \begin{cases} R_e [\partial \mathbf{v}_*/\partial t_* + (\mathbf{v}_* \cdot \nabla) \mathbf{v}_* + \nabla p_*] & \text{Velocity Equation} \\ -\nabla \cdot [(\mathbf{v}_* \cdot \nabla) \mathbf{v}_*] & \text{Pressure Equation} \end{cases} \quad (12)$$

The pressure equation is obtained by introducing the continuity equation into the divergence of the momentum equation. Note that in the velocity equation, Φ and F are vectors with two (for 2-D and axi-symmetric problems) or three (for 3-D problems) components. We will now drop the * prefix in the dimensionless variables, for convenience.

For most flow problems the boundary conditions will generally consist of three types:

- **Dirichlet Boundary** (Γ_Φ)

$$\Phi = \Phi_b$$

- **Neumann Boundary** (Γ_Q)

$$Q = \frac{\partial \Phi}{\partial n} = Q_b$$

where $\partial \Phi / \partial n = \nabla \Phi \cdot \mathbf{n}$, and \mathbf{n} is the unit vector normal to the boundary.

- **Mixed** (Γ_M)

$$\zeta(\Phi, \nabla \Phi, \mathbf{x}, t) = 0$$

where ζ is some specified function. A free-surface will be an example of the third. In most iterative schemes it is usual to recast the *Mixed* boundary condition in the form of either the *Dirichlet* or the *Neumann* types.

BOUNDARY INTEGRAL EQUATIONS

We will use equation (10) as the representative PDE in developing the integral equations. If fictitious sources of strength w are distributed around Γ , equation (10) can be converted into the integral expression (see *Jaswon & Symm [1977]*):

$$\Phi(\mathbf{x}) = \int_{\Gamma} w(\mathbf{x}')g(\mathbf{x}, \mathbf{x}') d\mathbf{x}' + \int_{\Omega} F(\mathbf{x}'')g(\mathbf{x}, \mathbf{x}'') d\mathbf{x}'' \quad (13)$$

where g is the free-space Green's function which must satisfy:

$$\nabla^2 g(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x}, \mathbf{x}') \quad (14)$$

where δ is the Dirac delta function applied at a point \mathbf{x}' and felt at \mathbf{x} . The closed form solution to equation (14) is (*Greenberg [1971]*):

$$g(\mathbf{x}, \mathbf{x}') = \begin{cases} \ln r/2\pi & \text{in two-dimensions} \\ 1/(4\pi r) & \text{in three-dimensions} \end{cases} \quad (15)$$

in which $r = |\mathbf{x} - \mathbf{x}'|$. The last term in equation (13) represents a domain integral. To convert this into an integration on the boundary we introduce the *Dual Reciprocal* concept (*Cheng et al., [1993]*).

DUAL RECIPROCAL TECHNIQUE

Consider n_T points on Γ and in Ω . We introduce a family of coordinate functions $M_j(\mathbf{x})$ ($j = 1, 2, \dots, n_T$) such that:

$$F(\mathbf{x}) \approx \sum_{j=1}^{n_T} \beta_j M_j(\mathbf{x}) \quad (16)$$

where β_j are coefficients to be determined by collocation. We assume for each function $M_j(\mathbf{x})$, there exists an associated function $\Psi_j(\mathbf{x})$ such that:

$$\nabla^2 \Psi_j(\mathbf{x}) = M_j(\mathbf{x}) \quad (17)$$

It can be shown (*Cheng & Ouazar [1993]*) that for a two-dimensional problem for which $M_j = x^m y^n$ the function Ψ_j is given by:

$$\Psi_j = \begin{cases} \sum_{k=1}^{\lfloor \frac{m+2}{2} \rfloor} (-1)^{k-1} \frac{m!n!x^{m+2k}y^{n-2k+2}}{(m+2k)!(n-2k+2)!} & m \geq n \\ \sum_{k=1}^{\lfloor \frac{m+2}{2} \rfloor} (-1)^{k-1} \frac{m!n!x^{m+2-k}y^{n+2k}}{(m-2k+2)!(n+2k)!} & m < n \end{cases} \quad (18)$$

where the square brackets in the upper limit of the summation denote the integer part of the argument. Solutions for other possible families of coordinate functions are presented in Table 1.

Table 1			
\mathcal{L}	Dim.	M_j	Ψ_j
∇^2	2D	$\cos(nx) \cos(my)$	$\frac{-M_j}{(n^2+m^2)}$
∇^2	Axi	$K_0(nr) \cos(kz)$	$\frac{-M_j}{(n^2+k^2)}$
∇^2	3D	$\cos(nx) \cos(my) \cos(kz)$	$\frac{-M_j}{(n^2+m^2+k^2)}$
∇^2	2D	$e^{(nx+my)}$	$\frac{M_j}{(n^2+m^2)}$
∇^2	Axi	$K_0(nr)e^{kz}$	$\frac{M_j}{(n^2+k^2)}$
∇^2	3D	$e^{(nx+my+kz)}$	$\frac{M_j}{(n^2+m^2+k^2)}$
$\nabla^2 - \lambda^2$	2D	$\cos(nx) \cos(my)$	$\frac{-M_j}{(n^2+m^2-\lambda^2)}$
$\nabla^2 - \lambda^2$	Axi	$K_0(nr) \cos(kz)$	$\frac{-M_j}{(n^2+k^2-\lambda^2)}$
$\nabla^2 - \lambda^2$	3D	$\cos(nx) \cos(my) \cos(kz)$	$\frac{-M_j}{(n^2+m^2+k^2-\lambda^2)}$
$\nabla^2 - \lambda^2$	2D	$e^{(nx+my)}$	$\frac{M_j}{(n^2+m^2-\lambda^2)}$
$\nabla^2 - \lambda^2$	Axi	$K_0(nr)e^{kz}$	$\frac{M_j}{(n^2+k^2-\lambda^2)}$
$\nabla^2 - \lambda^2$	3D	$e^{(nx+my+kz)}$	$\frac{M_j}{(n^2+m^2+k^2-\lambda^2)}$

in which K_0 is the zeroth order modified Bessel function of the first kind. When equations (16) and (17) are used in (10), and we distribute the fictitious sources on Γ we can obtain the 'pure' boundary integral equation:

$$\Phi(\mathbf{x}) = \int_{\Gamma} w(\mathbf{x}')g(\mathbf{x}, \mathbf{x}') d\mathbf{x}' + \sum_{j=1}^{n_T} \beta_j \Psi_j(\mathbf{x}) \quad (19)$$

An expression for the gradient of Φ , which is required in equation (12) can be obtained from equation (19) as:

$$\nabla \Phi(\mathbf{x}) = \int_{\Gamma} w(\mathbf{x}')\nabla g(\mathbf{x}, \mathbf{x}') d\mathbf{x}' + \sum_{j=1}^{n_T} \beta_j \nabla \Psi_j(\mathbf{x}) \quad (20)$$

The normal derivative $\partial\Phi/\partial n = \nabla\Phi \cdot \mathbf{n}$ is given by:

$$\frac{\partial\Phi}{\partial n}(\mathbf{x}) = \int_{\Gamma} w(\mathbf{x}') \frac{\partial g}{\partial n}(\mathbf{x}, \mathbf{x}') d\mathbf{x}' + \sum_{j=1}^{n_T} \beta_j \frac{\partial\Psi_j}{\partial n}(\mathbf{x}) \quad (21)$$

Assuming β_j ($j = 1, 2, \dots, n_T$) are known, the only unknown in equations (19) and/or (21) is the source strength distribution w on Γ . The trick is to start with a trial distribution of $F(\mathbf{x})$ and to obtain the coefficients β_j ($j = 1, 2, \dots, n_T$) by collocation using equation (16). When applied to all n_T selected points the result of the collocation is the matrix equation:

$$\sum_{j=1}^{n_T} M_{ij} \beta_j = F_i \quad i = 1, 2, \dots, n_T \quad (22)$$

where $M_{ij} = M_j(\mathbf{x}_i)$ and $F_i = F(\mathbf{x}_i)$. The matrix system (22) is also expressible in the form:

$$\mathbf{M}\beta = \mathbf{F} \quad (23)$$

which can be inverted to give:

$$\beta = \mathbf{M}^{-1}\mathbf{F} \quad (24)$$

Once β has been determined, equations (19) and/or (21) are then combined with the prescribed boundary conditions and solved for w on Γ . A better estimate of \mathbf{F} is then obtained by using equations (19) and (20) in (12). The solution process continues until a specified convergence criterion is satisfied.

DISCRETIZATION

We subdivide the boundary into n_b elements. Let $N_k(\mathbf{x})$ ($k = 1, 2, \dots, n_b$) represent the boundary shape functions describing the distribution of w on Γ . By selecting each of the n_b boundary points as successive origins of integration, equations (19) and (21) can be assembled into the system:

$$\sum_{k=1}^{n_b} a_{ik} w_k = b_i \quad i = 1, 2, \dots, n_b \quad (25)$$

where

$$a_{ik} = \begin{cases} \int_{\Gamma_k} N_k(\mathbf{x}') g(\mathbf{x}', \mathbf{x}_i) d\mathbf{x}' & \mathbf{x}_i \in \Gamma_{\phi} \\ \int_{\Gamma_k} N_k(\mathbf{x}') \partial g / \partial n(\mathbf{x}', \mathbf{x}_i) d\mathbf{x}' & \mathbf{x}_i \in \Gamma_Q \end{cases} \quad (26)$$

$$b_i = \begin{cases} \Phi(\mathbf{x}_i) - \sum_{j=1}^{n_T} \beta_j \Psi_{ij} & \mathbf{x}_i \in \Gamma_{\phi} \\ \partial\Phi/\partial n(\mathbf{x}_i) - \sum_{j=1}^{n_T} \beta_j \partial\Psi_{ij}/\partial n & \mathbf{x}_i \in \Gamma_Q \end{cases} \quad (27)$$

Therefore, we have n_b equations to determine w_k ($k = 1, 2, \dots, n_b$). Symbolically equation (25) can be written in the alternative form:

$$\mathbf{a}\mathbf{w} = \mathbf{b} \quad (28)$$

which can be inverted to give:

$$\mathbf{w} = \mathbf{a}^{-1}\mathbf{b} \quad (29)$$

The whole process boils down to the iterative solution of equations (24) and (29), with repeated updating of \mathbf{F} using (12). At each time level t , the iterative steps are:

1. Start with a trial \mathbf{F} (i.e., F_i values for $i = 1, 2, \dots, n_T$).

2. Obtain β from equation (24).
3. Obtain \mathbf{w} using equation (29).
4. Use discretized forms of equations (19) and (20) to compute Φ , $\nabla\Phi$ at all n_T points. This provides a better estimate for \mathbf{F} .
5. Go back to Step 2 if convergence condition is still unsatisfied.

Note that the matrix inversions in equations (24) and (29) need only be performed once, for fixed boundary problems. The vectors \mathbf{w} and β are the quantities whose values change during the iterative process. Once convergence is reached, equations (19) and (20) can be used routinely to obtain $\Phi = (\mathbf{v}_*, p_*)$ or the gradient at any point (\mathbf{x}) of interest.

Treatment of Time Derivatives

We now need to address the treatment of the local acceleration term $\partial\mathbf{v}/\partial t$ as occurs when equation (11) is written for the velocity. We discuss two efficient approaches for handling this term. The first is based on the use of a time-dependent fundamental solution. The second utilizes a simple time-marching procedure.

Time-dependent Fundamental Solution

Equation (11), written for the velocity (*i.e.*, $\Phi = \mathbf{v}$), can be re-arranged into the alternative form

$$\mathcal{L}\Phi(\mathbf{x}, t) = F(\mathbf{x}, t) \quad (30)$$

where

$$\mathcal{L} = \nabla^2(\cdot) - R_c \frac{\partial(\cdot)}{\partial t}$$

$$F = R_c [(\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p]$$

The boundary integral equation in this case is

$$\Phi(\mathbf{x}, t) = \int_0^t \int_V w(\mathbf{x}', t') g(\mathbf{x}, t, \mathbf{x}', t') d\mathbf{x}' dt' + \sum_{j=1}^{n_T} \beta_j \Psi_j(\mathbf{x}, t) \quad (31)$$

where the functions g and Ψ_j must respectively satisfy the following PDEs

$$\mathcal{L}g(\mathbf{x}, t, \mathbf{x}', t') = \delta(\mathbf{x}, t, \mathbf{x}', t') \quad (32)$$

$$\mathcal{L}\Psi(\mathbf{x}, t) = M_j(\mathbf{x}, t) \quad (33)$$

The closed-form solution for (32) is (see *Greenberg* [1971])

$$g(\mathbf{x}, t, \mathbf{x}', t') = \begin{cases} \frac{H(t-t')}{(t-t')} \exp\left\{-\frac{R_c r^2}{4(t-t')}\right\} & \text{in two-dimensions} \\ \frac{\sqrt{R_c} H(t-t')}{(t-t')^{3/2}} \exp\left\{-\frac{R_c r^2}{4(t-t')}\right\} & \text{in three-dimensions} \end{cases} \quad (34)$$

where H is the Heaviside function.

The extra computational effort required here is the time integration. $\int_0^t(\cdot)$, at each time level. However, this approach has the major advantage that no difference approximation is introduced in the evaluation of the time derivative, and the exact time-dependent fundamental solution is utilized in the integral equation.

Time-marching Procedure

In this approach we assume the time derivative can be approximated by the difference equation

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\mathbf{v} - \mathbf{v}_0}{\Delta t}$$

where \mathbf{v}_0 is the velocity at a previous time level. The velocity equation (30) is still valid but the differential operator \mathcal{L} and forcing function F now become

$$\begin{aligned} \mathcal{L} &= \nabla^2(\cdot) - \frac{R_e}{\Delta t}(\cdot) \\ F &= R_e \left[-\frac{\mathbf{v}_0}{\Delta t} + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0 + \nabla p \right] \end{aligned}$$

The boundary integral equation in this case is

$$\Phi(\mathbf{x}, t) = \int_{\Gamma} w(\mathbf{x}', t') g(\mathbf{x}, \mathbf{x}') d\mathbf{x}' + \sum_{j=1}^{n_T} \beta_j \Psi_j(\mathbf{x}, t) \quad (35)$$

where the functions g and Ψ_j must respectively satisfy

$$\mathcal{L}g(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x}, \mathbf{x}') \quad (36)$$

$$\mathcal{L}\Psi(\mathbf{x}, t) = M_j(\mathbf{x}, t) \quad (37)$$

It is easily shown that

$$g(\mathbf{x}, \mathbf{x}') = \begin{cases} K_0 \left(\sqrt{\frac{R_e}{\Delta t}} r \right) & \text{in two-dimensions} \\ \frac{1}{r} \exp \left\{ - \left(\sqrt{\frac{R_e}{\Delta t}} r \right) \right\} & \text{in three-dimensions} \end{cases} \quad (38)$$

The time-marching approach has the obvious advantage of not requiring an explicit time integration, as in the first method. Furthermore, for fixed boundary problems, the free-space function g need not be calculated at each time level. The iterative scheme can now be replaced by the time-marching process. However, the presence of Δt and R_e in the argument of the Green's function creates the need for extreme caution in the numerical evaluation of g . As the time step is reduced to improve accuracy, or as the flow moves away from the laminar regime, the numerical value of g reduces very rapidly.

CONCLUSIONS

New concepts in boundary element modeling provide excellent approaches for solving convective transport problems. A formal determination of the advantages inherent in the new DRBEM formulation can easily be accomplished. In general, we expect optimum n_T (=total number of collocation points selected to evaluate coefficients β) to be of the same order as the number, n_b , of boundary elements. Therefore, the largest matrix size in BEM will be of order $n_b \times n_b = n_b^2$. By comparison, the domain techniques, because of the need to discretize the entire region, will produce a global matrix size of order $n_b^2 \times n_b^2 = n_b^4$. The advantage in terms of storage requirements is obvious. Moreover, the much reduced size of the global matrix has a more pronounced advantage in total CPU time.

For example, in a 2-D flow problem discretized with 100 boundary elements, the matrix size using the domain methods (if no special consideration is given to matrix bandedness) will be 10^4 larger than that of BEM. Even with the sparseness of the global matrix taken into account in the domain methods, a boundary element approach still enjoys a size advantage proportional to the bandwidth of the matrix. The computational advantage in 3-D is more significant because of the much increased number of meshes in the domain techniques.

These computational advantages are key to effective modeling of convective flows. The compactness of BEM matrices allows for a greater freedom to experiment, even on computers of average memory. DRBEM provides an excellent platform for optimizing system geometry.

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